JOURNAL OF APPROXIMATION THEORY 37, 137–146 (1983)

Inequalities for Weighted Polynomials

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Communicated by Oved Shisha

Received September 12, 1980; revised March 18, 1982

IN MEMORY OF GÉZA FREUD

1. INTRODUCTION AND STATEMENT OF RESULTS

Let $q_n(x)$ be a polynomial of degree n. The well-known inequality of Markov states that

$$\|q_n'(x)\|_{L_\infty[-1,1]} \leqslant n^2 \|q_n(x)\|_{L_\infty[-1,1]}$$

(cf. Timan [1, p. 218]). It is natural to conjecture that similar inequalities may hold for weighted polynomials on unbounded intervals, and indeed Milne [2] obtained a result which by a trivial change of variable is equivalent to the following inequality:

$$\|[\exp(-x^2/2)\, q_n(x)]'\|_{L_\infty(-\infty,\infty)} \leqslant K_1 n^{1/2} \|\exp(-x^2/2)\, q_n(x)\|_{L_\infty(-\infty,\infty)}, \quad (1)$$

where K_1 is a constant independent of n and of $q_n(x)$. Almost 40 years after the publication of Milne's paper, G. Freud showed that for $1 \leqslant p \leqslant \infty$,

$$\|\exp(-x^2/2)\, q_n'(x)\|_{L_p(-\infty,\infty)} \leqslant K_2 n^{1/2} \|\exp(-x^2/2)\, q_n(x)\|_{L_p(-\infty,\infty)} \quad (2)$$

(cf. [3, Theorem 1], and

$$\|\exp(-x^2/2)\, q_n(x)\|_{L_1(-\infty,\infty)} \leqslant K_3 \int_{-4\sqrt{n}}^{4\sqrt{n}} \exp(-x^2/2)\, |q_n(x)|\, dx \quad (3)$$

(cf. [3, Lemma 1]). Using (3), he also showed that

$$\|x \exp(-x^2/2)\, q_n(x)\|_{L_1(-\infty,\infty)} \leqslant K_4 n^{1/2} \|\exp(-x^2/2)\, q_n(x)\|_{L_1(-\infty,\infty)} \quad (4)$$

(cf. [3; Lemma 2]).

* The author is grateful to S. Kwapień and the referee for their very helpful comments.

0021-9045/83/020137-10$03.00/0

It is easy to show that (2) and (4) imply (1); thus Freud's research can be considered to be a generalization of Milne's, although the former was apparently unaware of the work of the latter.

The purpose of this paper is to prove the following propositions:

THEOREM 1. (a) *Let* $0 < r < \infty$, $1 \leqslant p \leqslant \infty$, *and assume that* $n > 0$ *is an integer. Then there is a constant* a, *independent of* n, *such that*

$$\| |x|^r \exp(-x^2/2)\, q_n(x)\|_{L_p(-\infty,\infty)} \leqslant a n^{r/2} \|\exp(-x^2/2)\, q_n(x)\|_{L_p(-\infty,\infty)}. \quad (5)$$

(b) *The above inequality is optimal in the sense that the constant* a *cannot be replaced by a sequence* $\{A_n\}$ *that converges to zero as* n *tends to infinity.*

THEOREM 2. (a) *Let* $s > 0$ *and* $n > 0$ *be integers, and assume that* $1 \leqslant p \leqslant \infty$. *Then there is a constant* b, *independent of* n, *such that*

$$\|[\exp(-x^2/2)\, q_n(x)]^{(s)}\|_{L_p(-\infty,\infty)} \leqslant b n^{s/2} \|\exp(-x^2/2)\, q_n(x)\|_{L_p(-\infty,\infty)}. \quad (6)$$

(b) *The above inequality is optimal* (*in the sense of Theorem* 1(b)).

THEOREM 3. *Let* $1 \leqslant p, p_1 \leqslant \infty$, *and assume that* $n > 0$ *is an integer. Then there is a constant* c, *independent of* n, p, *and* p_1, *such that*

$$\|\exp(-x^2/2\, q_n(x)\|_{L_p(-\infty,\infty)} \leqslant c n^{|1/(2p) - 1/(2p_1)|} \|\exp(-x^2/2)\, q_n(x)\|_{L_{p_1}(-\infty,\infty)}. \quad (7)$$

THEOREM 4. *Let* $1 \leqslant p \leqslant \infty$, *and assume that* $n > 0$ *is an integer. Then there is a constant* d, *independent of* n *and* p, *such that*

$$\|\exp(-x^2/2)\, q_n(x)\|_{L_p(-\infty,\infty)} \leqslant d \|\exp(-x^2/2)\, q_n(x)\|_{L_p(-4\sqrt{n},4\sqrt{n})}. \quad (8)$$

Remarks. Note that (5), (6), and (8) generalize (4), (1), and (3), whereas (7) is similar to the results of Timan for polynomials and bounded intervals (see [1, p. 236, and also p. 229]). The case $p = \infty$ in Theorem 4 and some generalizations of it are known [4, 5].

2. Proofs

Let $w(x)=\exp(-x^2/2)$, *and let* $\| \ \|_p$ *stand for* $\| \ \|_{L_p(-\infty,\infty)}$.

Proof of Theorem 1(a). We first show that if $q_n(x)$ is a polynomial of degree n, and r is nonnegative integer, there is a constant a_1, such that

$$\|x^r w(x)\, q_n(x)\|_\infty \leqslant a_1 n^{r/2} \|w(x)\, q_n(x)\|_\infty. \tag{9}$$

Note that for $r=0$, the assertion is trivial. Assume that (9) holds. Since

$$[w(x)\, x^r q_n(x)]' = w(x)[x^r q_n(x)]' - w(x)\, x^{r+1} q_n(x), \tag{10}$$

we have:

$$\|w(x)\, x^{r+1} q_n(x)\|_\infty \leqslant \|[w(x)\, x^r q_n(x)]'\|_\infty + \|w(x)[x^r q_n(x)]'\|_\infty,$$

and the proof of the inductive step follows by applying (1) and (9) to the first term in the right hand of the preceding inequality, and (2) and (9) to the second term.

We now show that (9) holds if r is any positive real number. To see this, note that for fixed n, q_n, and x, $|n^{-1/2}x|^r\,|w(x)\, q_n(x)|$ is a convex function of r; it is thus readily seen that for fixed n and q_n, $h(r)=n^{-r/2}\||x|^r w(x)\, q_n(x)\|_\infty/\|w(x)\, q_n(x)\|_\infty$ is convex on $[0,\infty)$. Thus if $r>0$, and r_1 is an integer larger than r, we know that $h(r)\leqslant \max[h(0), h(r_1)]$, and the conclusion follows by noting that since (9) holds for integers, $h(0)$ and $h(r_1)$ are numbers independent of n and of $q_n(x)$.

We now prove (5). In view of (9), it suffices to assume that $1\leqslant p<\infty$. We first need to prove Theorem 4 for $p<\infty$. From [3, p. 571, line 2] we see that if $|x|\geqslant 4\sqrt{n}$, and $1\leqslant p<\infty$

$$|w(x)\, q_n(x)| \leqslant a_2 \|w(x)\, q_n(x)\|_1 \exp(-a_3 n), \tag{11}$$

where a_2 and a_3 are positive and independent of n. Thus,

$$\int_{-n}^{-4\sqrt{n}} |w(x)\, q_n(x)|^p\, dx + \int_{4\sqrt{n}}^{n} |w(x)\, q_n(x)|^p\, dx$$

$$\leqslant a_2^p(\|w(x)\, q_n(x)\|_1)^p \exp(-a_4\, pn),$$

where a_4 is positive and independent of n and p.

From [3, pp. 570–571, (3) and (5)] we infer that if $|x|\geqslant n$, then

$$|w(x)\, q_n(x)| \leqslant a_5 n \|w(x)\, q_n(x)\|_1 \exp(-x^2/16n).$$

Thus,

$$\int_{|x|\geqslant n} |w(x)\,q_n(x)|^p\,dx \leqslant a_5^p n^p (\|w(x)\,q_n(x)\|_1)^p \int_n^\infty \exp(-px^2/16n)\,dx$$

$$= a_5^p n^p (\|w(x)\,q_n(x)\|_1)^p \int_n^\infty x^{-1}[x\exp(-px^2/16n)]\,dx$$

$$\leqslant a_5^p (\|w(x)\,q_n(x)\|_1)^p\, a_6 \exp(-a_7\,pn),$$

where a_5, a_6, and a_7 are independent of n and p. Setting $a_8 = \max(a_2, a_5 a_6, a_5)$, and $a_9 = \min(a_4, a_7)$, we conclude from the two preceding inequalities that

$$\left[\int_{|x|\geqslant 4\sqrt{n}} |w(x)\,q_n(x)|^p\,dx\right]^{1/p} \leqslant a_8 \|w(x)\,q_n(x)\|_1 \exp(-a_9 n). \tag{12}$$

From (3), dividing and multiplying by $1+x^2$ and applying Hölder's inequality, we see that $\|w(x)\,q_n(x)\|_1 \leqslant \pi K_3(1+16n)\,\|w(x)\,q_n(x)\|_p$. Since

$$\|w(x)\,q_n(x)\|_p \leqslant \left[\int_{|x|\leqslant 4\sqrt{n}} |w(x)\,q_n(x)|^p\,dx\right]^{1/p} + \left[\int_{|x|\geqslant 4\sqrt{n}} |w(x)\,q_n(x)|^p\,dx\right]^{1/p},$$

we conclude from (12) that there is a constant a_{10}, independent of n and p, such that

$$\|w(x)\,q_n(x)\|_p \leqslant a_{10} \left[\int_{-4\sqrt{n}}^{4\sqrt{n}} |w(x)\,q_n(x)|^p\,dx\right]^{1/p}. \tag{13}$$

Assume now that r is an integer; thus $x^r q_n(x)$ is a polynomial of degree $n+r$, and from (13) we have

$$\|w(x)\,x^r q_n(x)\|_p \leqslant a_{10} \left[\int_{-4\sqrt{n+r}}^{4\sqrt{n+r}} |w(x)\,x^r q_n(x)|^p\,dx\right]^{1/p}$$

$$\leqslant a_{11} n^{r/2} \left[\int_{-4\sqrt{n+r}}^{4\sqrt{n+r}} |w(x)\,q_n(x)|^p\,dx\right]^{1/p}$$

$$\leqslant a_{11} n^{r/2} \|w(x)\,q_n(x)\|_p,$$

and (5) follows. We have therefore proved (5) for integral r.

To prove (5) for any $r \geqslant 0$, note that for fixed x, $[n^{-1/2}|x|^p]^r\,|\,|w(x)\,q_n(x)|^p/(\|w(x)\,q_n(x)\|_p)^p$ is a convex function of r; thus also Riemann sums of this function are convex, and we infer that

$n^{-r/2}(\||x|^r w(x)\, q_n(x)\|_p)^p/(\|w(x)\, q_n(x)\|_p)^p$ is a convex function of r. The conclusion now follows as in the proof of (9).

(b) Assume $1 \leqslant p < \infty$, and let $q_n(x) = x^n$; thus

$$\begin{aligned}(\||x|^r w(x)\, q_n(x)\|_p)^p &= \int_R |x|^{(r+n)p} \exp[-\tfrac{1}{2}(\sqrt{p}x)^2]\, dx \\ &= 2\int_0^\infty |x|^{(r+n)p} \exp[-\tfrac{1}{2}(\sqrt{p}\, x)^2]\, dx \\ &= (2/p)^{[(r+n)p+1]/2} \int_0^\infty u^{[(r+n)p-1]/2} \exp(-u)\, du \\ &= (2/p)^{[(r+n)p+1]/2}\, \Gamma[\tfrac{1}{2}(rp+np+1)].\end{aligned}$$

Similarly, $(\|w(x)\, q_n(x)\|_p)^p = (2/p)^{(np+1)/2}\, \Gamma[\frac{1}{2}(np+1)]$. Applying now Stirling's formula (cf. Lebedev [6, p. 12 (1.4.25)]), we see that $(\||x|^r w(x)\, q_n(x)\|_p)^p \geqslant a_{12} n^{rp/2}(\|w(x)\, q_n(x)\|_p)^p$, and the conclusion follows. To prove the asertion for $p = \infty$, we use elementary calculus to conclude that $\||x^r w(x)\, q_n(x)\|_\infty = \exp[-\frac{1}{2}(n+r)](n+r)^{(n+r)/2}$, and $\|w(x)\, q_n\|_\infty = \exp[-(\frac{1}{2})\, n]\, n^{n/2}$, whence the conclusion readily follows. Q.E.D.

Proof of Theorem 2. (a) Assume first that $p = \infty$. We proceed by induction. For $s = 1$, (6) reduces to (1). Assume that (6) holds; applying (10), then (6), and finally (1), we obtain:

$$\begin{aligned}&\|[w(x)\, q_n(x)]^{(s+1)}\|_\infty = \|[w(x)(q_n'(x) - xq_n(x))]^{(s)}\|_\infty \\ &\quad\leqslant b(n+1)^{s/2}\, \|w(x)[q_n'(x) - xq_n(x)]\|_\infty = b(n+1)^{s/2}\, \|[w(x)\, q_n(x)]'\|_\infty \\ &\quad\leqslant bK_1(n+1)^{s/2}\, n^{1/2}\, \|w(x)\, q_n(x)\|_\infty \leqslant b_1 n^{(s+1)/2}\, \|w(x)\, q_n(x)\|_\infty,\end{aligned}$$

and the conclusion follows.

Assume now that $1 \leqslant p < \infty$. We again proceed by induction. The assertion is trivial for $s = 0$. Assume that (6) holds. Applying (10), (6), (2), and finally (4), we have:

$$\begin{aligned}\|[w(x)\, q_n(x)]^{(s+1)}\|_p &\leqslant \|[w(x)\, q_n'(x)]^{(s)}\|_p + \|[xw(x)\, q_n(x)]^{(s)}\|_p \\ &\leqslant bn^{s/2}\, \|w(x)\, q_n'(x)\|_p + b(n+1)^{s/2}\, \|xw(x)\, q_n(x)\|_p \\ &\leqslant bK_2 n^{(s+1)/2}\, \|w(x)\, q_n(x)\|_p + b_2 n^{s/2}\, \|xw(x)\, q_n(x)\|_p \\ &\leqslant b_3 n^{(s+1)/2}\, \|w(x)\, q_n(x)\|_p + b_2 K_4 n^{(s+1)/2}\, \|w(x)\, q_n(x)\|_p,\end{aligned}$$

and the conclusion follows.

(b) From the Rodrigues formula (cf. Szegö [7, p. 106, (5.5.3)]) and the chain rule, we readily infer that

$$w(x)\, H_n(x/\sqrt{2}) = (-1)^n\, 2^{n/2}[w(x)]^{(n)}.$$

Thus,

$$[w(x)\, H_n(x/\sqrt{2})]^{(s)} = (-1)^n\, 2^{n/2}[w(x)]^{(n+s)} = (-1)^s\, 2^{-s/2} w(x)\, H_{n+s}(x/\sqrt{2}). \tag{14}$$

From [4; p. 94 Exercise 6], we see by a change of variable that

$$\int_R |w(x)\, H_n(x/\sqrt{2})|^2\, dx = \int_R \exp(-x^2)\, H_n^2(x/\sqrt{2})\, dx = \sqrt{2}\int_R \exp(-2x^2)\, H_n^2(x)\, dx = 2^{n-1/2}\Gamma(n+1/2), \tag{15}$$

and

$$\int_R |w(x)\, H_{n+s}(x/\sqrt{2})|^2\, dx = 2^{n+s-1/2}\Gamma(n+s+\tfrac{1}{2}).$$

Thus from (14) we see that

$$\int_R |[w(x)\, H_n(x/\sqrt{2})]^{(s)}|^2\, dx = 2^{n-1/2}\Gamma(n+s+\tfrac{1}{2}); \tag{16}$$

combining (15) and (16) and applying Stirling's formula, we conclude that

$$\int_R |[w(x)\, H_n(x/\sqrt{2})]^{(s)}|^2\, dx \geqslant b_3 n^s \int_R |w(x)\, H_n(x/\sqrt{2})|^2\, dx,$$

i.e.,

$$\|[w(x)\, H_n(x/\sqrt{2})]^{(s)}\|_2 \geqslant b_4 n^{s/2}\, \|w(x)\, H_n(x/\sqrt{2})\|_2. \tag{17}$$

We have therefore proved the assertion for $p=2$. To prove the assertion for all values of $p \geqslant 1$, we shall use the Riesz–Thorin interpolation theorem (cf. Zygmund [8, Vol. II, p. 95]; this method is briefly outlined in [3, p. 572]). Let $V_n(f; t)$ be defined as in Freud [9, p. 371, (7)]. As remarked in that paper, if $\gamma = [n/2]$, and if $f(x)$ is a polynomial of degree γ, then $V_n(f; t) = f(t)$ identically.

From [9, p. 371, (8) and (9)], and the Riesz–Thorin theorem, we readily infer that for all p such that $1 \leqslant p \leqslant \infty$, and every function $f(x)$ such that $w(x) f(x)$ is p-integrable,

$$\|w(x)\, V_n(f; x)\|_p \leqslant b_5\, \|w(x) f(x)\|_p. \tag{18}$$

Assume now that for some s and p there is a sequence M_n, converging to zero, such that for every polynomial $q_n(x)$ of degree n,

$$\|[w(x)\,q_n(x)]^{(s)}\|_p \leqslant M_n n^{s/2}\,\|w(x)\,q_n(x)\|_p. \tag{19}$$

Define the linear operator $T_{n,s}(f)$ by

$$[T_{n,s}(f)](x) = [w(x)\,V_{2n}(f;x)]^{(s)}.$$

Let $M_n' = b_5 M_n$; applying (19) and then (18) we have

$$\begin{aligned}\|T_{n,s}(f)\|_p = \|[w(x)\,V_{2n}(f;x)]^{(s)}\|_p &\leqslant M_{2n}(2n)^{s/2}\,\|w(x)\,V_{2n}(f;x)\|_p\\ &\leqslant M_{2n}'(2n)^{s/2}\,\|w(x)f(x)\|_p.\end{aligned} \tag{20}$$

If $p>2$, let us choose a number q from the interval $(1,2)$, whereas if $p<2$, let $q>2$; in either case, $\frac{1}{2} = \theta p^{-1} + (1-\theta)\,q^{-1}$, where $0<\theta<1$. From (6) and (18),

$$\begin{aligned}\|T_{n,s}(f)\|_q = \|[w(x)\,V_{2n}(f;x)]^{(s)}\|_q &\leqslant b(2n)^{s/2}\,\|w(x)\,V_{2n}(f;x)\|_q\\ &\leqslant b_6(2n)^{s/2}\,\|w(x)f(x)\|_q.\end{aligned} \tag{21}$$

Applying (20), (21), and the Riesz–Thorin theorem, we thus conclude that $\|T_{n,s}(f)\|_2 \leqslant b_6^{1-\theta}(2n)^{s/2}(M_{2n}')^\theta\,\|w(x)f(x)\|_2$, for every function $f(x)$ such that $w(x)f(x)$ is square-integrable. In particular,

$$\begin{aligned}\|[w(x)\,H_n(x/\sqrt{2})]^{(s)}\|_2 &= \|T_{n,r}[H_n(x/\sqrt{2})]\|_2\\ &\leqslant b_6^{1-\theta}(2n)^{s/2}(M_{2n}')^\theta\,\|w(x)\,H_n(x/\sqrt{2})\|_2.\end{aligned}$$

Since $(M_{2n}')^\theta$ converges to zero as $n\to\infty$ this contradicts (17), and the conclusion follows. Q.E.D.

Proof of Theorem 3. Assume that $p<p_1<\infty$. Applying Pólya and Szegö [10, p. 65, Problem 71], with

$$\varphi(x) = |x|^{p_1/p} \qquad \text{and} \qquad f(x) = |w(x)\,q_n(x)|^p,$$

we see that

$$\left[(8\sqrt{n})^{-1}\int_{-4\sqrt{n}}^{4\sqrt{n}} |w(x)\,q_n(x)|^p\,dx\right]^{p_1/p} \leqslant (8\sqrt{n})^{-1}\int_{-4\sqrt{n}}^{4\sqrt{n}} |w(x)\,q_n(x)|^{p_1}\,dx$$

i.e.,

$$\left[\int_{-4\sqrt{n}}^{4\sqrt{n}} |w(x)\,q_n(x)|^p\,dx\right]^{1/p} \leqslant (64n)^{(1/(2p)-1/(2p_1))}\left[\int_{-4\sqrt{n}}^{4\sqrt{n}} |w(x)\,q_n(x)|^{p_1}\,dx\right]^{1/p_1}.$$

Since clearly

$$\left[\int_{-4\sqrt{n}}^{4\sqrt{n}} |w(x)\,q_n(x)|^{p_1}\,dx\right]^{1/p_1} \leqslant \|w(x)\,q_n(x)\|_{p_1}, \tag{22}$$

the conclusion readily folows from (13). If $p_1 = \infty$, the conclusion follows from (22) and (13) by noting that

$$\lim_{r\to\infty} \left[\int_{-4\sqrt{n}}^{4\sqrt{n}} |w(x)\,q_n(x)|^r\right]^{1/r} = \sup |w(x)\,q_n(x)|,$$

where the supremum is taken on $[-4\sqrt{n}, 4\sqrt{n}]$ (cf. e.g., Cotlar and Cignoli [11, p. 286, Lemma 1.2.3]).

The remainder of the proof is carried out by adapting an argument of Timan (cf. [1, p. 236]).

Assume $p_1 < p = \infty$. Let x_0 be such that $\|w(x)\,q_n(x)\|_\infty = |w(x_0)\,q_n(x_0)|$. Applying the mean value theorem and (6) we have:

$$\begin{aligned} |w(x_0)\,q_n(x_0)| - |w(x)\,q_n(x)| &\leqslant |w(x)\,q_n(x) - w(x_0)\,q_n(x_0)| \\ &\leqslant |x-x_0|\,\|[w(x)\,q_n(x)]'\|_\infty \leqslant bn^{1/2}\,|x-x_0|\,\|w(x)\,q_n(x)\|_\infty. \end{aligned}$$

In view of the definition of x_0, it is therefore clear that for all real x

$$[1 - bn^{1/2}\,|x-x_0|]\,\|w(x)\,q_n(x)\|_\infty \leqslant |w(x)\,q_n(x)|. \tag{23}$$

Assume, e.g., that $x > x_0$, and let $b_n = x_0 + (bn^{1/2})^{-1}$; then

$$\int_{x_0}^{b_n} [1 - bn^{1/2}\,|x-x_0|]^{p_1}\,dx = [(bn^{1/2})(p_1+1)]^{-1}.$$

Raising both terms of (23) to the p_1th power and integrating, we thus conclude that

$$[(bn^{1/2})(p_1+1)]^{-1}(\|w(x)\,q_n(x)\|_\infty)^{p_1} \leqslant \int_{x_0}^{b_n} |w(x)\,q_n(x)|^{p_1}\,dx,$$

whence

$$\|w(x)\,q_n(x)\|_\infty \leqslant (p_1+1)^{1/p_1}(bn^{1/2})^{1/p_1}\,\|w(x)\,q_n(x)\|_{p_1},$$

i.e.,

$$\|w(x)\,q_n(x)\|_\infty \leqslant cn^{1/(2p_1)}\,\|w(x)\,q_n(x)\|_{p_1},$$

where the constant c is independent of n and p_1.

If $p_1 < p < \infty$, we have

$$(\| w(x)\, q_n(x)\|_p)^p = \int_R |w(x)\, q_n(x)|^p \, dx$$

$$= \int_R |w(x)\, q_n(x)|^{p_1} \, |w(x)\, q_n(x)|^{p-p_1} \, dx$$

$$\leqslant (\| w(x)\, q_n(x)\|_\infty)^{p-p_1} (\| w(x)\, q_n(x)\|_{p_1})^{p_1},$$

i.e.,

$$(\| w(x)\, q_n(x)\|_p)^p \leqslant (\| w(x)\, q_n(x)\|_\infty)^{p-p_1} (\| w(x)\, q_n(x)\|_{p_1})^{p_1}.$$

Thus

$$(\| w(x)\, q_n(x)\|_p)^p \leqslant (cn^{1/(2p_1)} \, \| w(x)\, q_n(x)\|_{p_1})^{p-p_1} (\| w(x)\, q_n(x)\|_{p_1})^{p_1},$$

whence the conclusion readily follows. Q.E.D.

Proof of Theorem 4. For $p < \infty$, the conclusion follows from (13). Assume therefore that $p = \infty$, and select $a > 0$ and r, $1 < r < \infty$, arbitrarily. Since clearly

$$\| w(x)\, q_n(x)\|_{L_r[-a,a]} \leqslant \| w(x)\, q_n(x)\|_{L_r},$$

we infer from (13) that

$$\| w(x)\, q_n(x)\|_{L_r[-a,a]} \leqslant a_{10} \, \| w(x)\, q_n(x)\|_{L_r[-4\sqrt{n},\, 4\sqrt{n}]}.$$

Since a_{10} does not depend on r, making $r \to \infty$ we see that

$$\| w(x)\, q_n(x)\|_{L_\infty[-a,a]} \leqslant a_{10} \, \| w(x)\, q_n(x)\|_{L_\infty[-4\sqrt{n},\, 4\sqrt{n}]},$$

and the conclusion follows by noting that

$$\| w(x)\, q_n(x)\|_\infty = \sup\{\| w(x)\, q_n(x)\|_{L_\infty[-a,a]},\, a > 0\}.$$

Q.E.D.

Remark. After writing this paper the author discovered that another proof of Theorem 2(a), with a more general weight function, was obtained by G. Freud in [12, p. 129, Theorem 2]. As part of a forthcoming article the autor will include the generalization of results of this paper, for Freud's weight function.

REFERENCES

1. A. F. TIMAN, "Theory of Approximation of Functions of a Real Variable," J. Berry, transl., Macmillan, New York, 1963.
2. W. E. MILNE, On the maximum absolute value of the derivative of $e^{-x^2}P_n(x)$, *Trans. Amer. Math. Soc.* **33** (1931), 143–146.
3. G. FREUD, On an inequality of Markov type, *Soviet Math. Dokl.* **12** (1971), 570–573. (Revised translation from *Dokl. Akad. Nauk SSSR* **197** (1971)).
4. G. FREUD, On two polynomial inequalities. I. *Acta Math. Acad. Sci. Hungar.* **22** (1971), 109–116.
5. G. P. NÉVAI, A certain inequality of G. Freud. (Russian), *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **16** (1973), 87–92.
6. N. N. LEBEDEV, "Special Functions and Their Applications," revised English ed. (R. A. Silverman, Ed. and transl.), Dover, New York, 1972.
7. G. SZEGÖ, "Orthogonal Polynomials," 4th ed., Amer. Math. Soc. Colloq. Publ. No. 23, Amer. Math. Soc., Providence, R. I., 1975.
8. A. ZYGMUND, "Trigonometric Series," Vol. I and II, 2nd ed., Cambridge Univ. Press, Cambridge, repr., 1968.
9. G. FREUD, On weighted approximation by polynomials on the real axis, *Soviet Math. Dokl.* **11** (1970), 370–372. (Translation from *Dokl. Akad. Nauk. SSSR* **191** (1970)).
10. G. PÓLYA AND G. SZEGÖ, "Problems and Theorems in Analysis," Vol. I, (D. Aeppli, transl.), Springer-Verlag, New York, 1976.
11. M. COTLAR AND N. CIGNOLI, "An Introduction to Functional Analysis," American Elsevier, New York, 1974.
12. G. FREUD, On direct and converse theorems in the theory of weighted polynomial approximation, *Math. Z* **126** (1972), 123–134.